Failure

- Failure means the component is not able to perform its function satisfactorily for which it is designed.
- S_{ys}, S_{yt} and S_{yc} denotes yield strength in Shear, Tension, and Compression respectively.
- In ductile materials, failure occurs at onset of plastic deformation i.e. yielding. Ductile materials are weak in shear i.e. $S_{yc} > S_{yt} > S_{ys}$. In this, $S_{yc} \approx S_{yt}$.
- S_{us}, S_{ut} and S_{uc} denotes ultimate strength in Shear, Tension, and Compression respectively.
- In brittle materials, failure occurs at fracture. Brittle materials are weak in tension and strong in compression i.e. $S_{uc} > S_{us} > S_{ut}$. In this, $S_{uc} = (3 \text{ to } 4)\, S_{ut}$

Miscellaneous

- **Strength Criterion:**

  Maximum stress induced ≤ Permissible stress

- **Factor of Safety (FOS):**

$$\textbf{FOS (N)} = \frac{\textbf{Failure Stress}}{\textbf{Permissible stress}}$$

- Failure stress/Maximum stress/Allowable stress/Ultimate stress
- Permissible stress/allowable stress/design stress/working stress

Theories of Failure (TOF)

- TOFs are used to determine safe dimensions of the components when it is subjected to combined stresses (i.e. under biaxial or triaxial state of stress) due to the various loads acting on it.
- TOFs are used in the design of following machine components:
    1. IC engine crankshaft
    2. Shafts used in power transmission
    3. Ceiling fan road
    4. Spindle of screw jack
    5. Bolted joints used under eccentric loading
    6. Welded joints used under eccentric loading
- TOFs are used in design of above machine components due to unavailability of failure stress under similar combined loading conditions.

Theories of Failure (TOF)

- TOFs are used to establish a relationship between stresses induced under combined loading conditions and the properties obtained from tension test (S_{yt} and S_{ut}).
- Strength criterion or all TOFs will give same result under uniaxial state of stress condition, hence TOFs are optional under such conditions. Also, it is easy to predict failure under uniaxial stress.
- Under biaxial and triaxial state of stress condition all TOFs will give different results hence appropriate TOF should be selected. In these conditions, it is difficult to predict failure.
- When major principal stress i.e. σ_1 is very large in comparison to other principal stresses [i.e. $\sigma_1 >>> \sigma_2$ and σ_3] then all TOF will give same result.

Various Theories of Failure (TOF)

1. Maximum Principal Stress Theory (M.P.S.T.) or Rankine Theory (for BRITTLE)
2. Maximum Shear Stress Theory (M.S.S.T.) or Guest & Tresca's theory (MORE SAFE but UNECONOMICAL) (for DUCTILE)
3. Maximum Principal Strain Theory (M.P.St.T.) or St. Venant's Theory
4. Total Strain Energy Theory (T.S.E.T.) or Haigh's Theory
5. Maximum Distortion Energy Theory (MDET) or Von Mises & Hencky Theory (SAFE and ECONOMICAL) (for DUCTILE)

Maximum Principal Stress Theory (M.P.S.T.)

- **Condition for safe design** $\sigma_1 \leq \frac{S_{yt}}{N}$ (or) $\frac{S_{ut}}{N}$
- **Condition for failure** $\sigma_1 > [S_{yt} (S_{yc})]$ (or) $[S_{ut} (S_{uc})]$
- Suitable for the safe design of machine components made of brittle materials under all loading conditions because brittle materials are weak in tension.
- Not suitable for safe deign of machine components made of ductile materials under every loading condition because ductile materials are weak in shear.
- But this theory is also applicable for ductile materials under following conditions:
    1. Uniaxial state of stress condition
    2. Biaxial state of stress condition when σ_1 and σ_2 are like in nature
    3. Hydrostatic stress condition ($\sigma_1 = \sigma_2 = \sigma_3 = \sigma$)

Maximum Shear Stress Theory (M.S.S.T.)

- MPST and MSST will give same result under biaxial state of stress when principal stresses are like in nature.
- MSST is not valid under Hydrostatic stress condition since abs $\tau_{max} = 0$.
- Suitable for ductile materials.

Condition for failure

$$\text{Abs } \tau_{max} > [S_{ys}] \text{ (or) } \frac{S_{yt}}{2}$$

Condition for safe design

$$\text{Abs } \tau_{max} \leq \frac{[S_{ys}]}{N} \text{ (or) } \frac{S_{yt}}{2N}$$

$$\text{larger of } \left[\left|\frac{\sigma_1-\sigma_2}{2}\right|, \left|\frac{\sigma_2-\sigma_3}{2}\right|, \left|\frac{\sigma_3-\sigma_1}{2}\right| \right] \leq \frac{S_{yt}}{2N}$$

$$\boxed{\text{larger of } \left[|\sigma_1-\sigma_2|, |\sigma_2-\sigma_3|, |\sigma_3-\sigma_1| \right] \leq \frac{S_{yt}}{N}}$$

for biaxial state of stress, $\sigma_3 = 0$

$$\left|\frac{\sigma_1}{2}\right| \text{ (or) } \left|\frac{\sigma_1-\sigma_2}{2}\right| \leq \frac{S_{yt}}{2N}$$

$$\boxed{\sigma_1 \leq \frac{S_{yt}}{N}}$$ [when σ_1 & σ_2 are like in nature]

$$\boxed{\sigma_1 - \sigma_2 \leq \frac{S_{yt}}{N}}$$ [when σ_1 & σ_2 are unlike in nature]

Maximum Principal Strain Theory (M.P.St.T.)

Condn for failure

$$\epsilon_1 > [\epsilon_{y.p.}]_{T.T.} \text{ or } \left[\frac{S_{yt}}{E}\right]$$

Condn for safe design

$$\epsilon_1 \leq [\epsilon_{y.p.}]_{T.T.} \text{ or } \frac{S_{yt}}{EN}$$

$$\frac{1}{E}\left[\sigma_1 - \mu(\sigma_2 + \sigma_3)\right] \leq \frac{S_{yt}}{EN}$$

$$\sigma_1 - \mu(\sigma_2 + \sigma_3) \leq \frac{S_{yt}}{N}$$

for biaxial state of stress, $\sigma_3 = 0$

$$\sigma_1 - \mu\sigma_2 \leq \frac{S_{yt}}{N}$$

Total Strain Energy Theory (T.S.E.T.)

Cond^n for failure,

$$\text{Total S.E./Vol.} > \left[(\text{S.E./Vol.})_{Y.P.}\right]_{T.T.}$$

Cond^n for safe design

$$\text{Total S.E./Vol.} \le \left[(\text{S.E./Vol.})_{Y.P.}\right]_{T.T.}$$

Under triaxial state of stress

$$\text{Total S.E./Vol.} = \frac{1}{2}\sigma_1\epsilon_1 + \frac{1}{2}\sigma_2\epsilon_2 + \frac{1}{2}\sigma_3\epsilon_3 \quad \text{--- (1)}$$

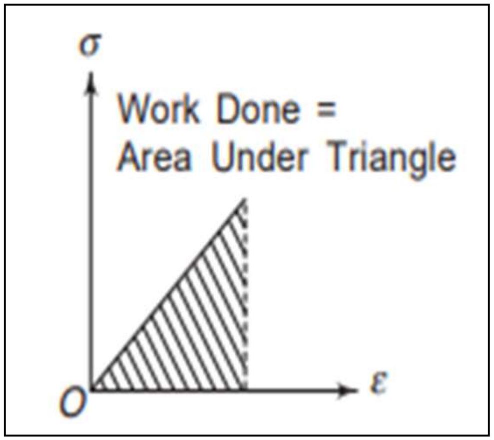
$$\epsilon_1 = \frac{1}{E}\left[\sigma_1 - \mu(\sigma_2+\sigma_3)\right]$$

$$\epsilon_2 = \frac{1}{E}\left[\sigma_2 - \mu(\sigma_1+\sigma_3)\right] \quad \text{--- (2)}$$

$$\epsilon_3 = \frac{1}{E}\left[\sigma_3 - \mu(\sigma_1+\sigma_2)\right]$$

Substitute eq^n (2) into (1),

$$\text{Total S.E./Vol.} = \frac{1}{2E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2 - 2\mu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\right] \quad \text{--- (3)}$$



For $\left[(\text{S.E./Vol.})_{Y.P.}\right]_{T.T.}$,

put $\sigma_2 = \sigma_3 = 0$ & $\sigma_1 = \sigma = \frac{S_{yt}}{N}$ in (3)

$$\left[(\text{S.E./Vol.})_{Y.P.}\right]_{T.T.} = \frac{\sigma_1^2}{2E} = \frac{1}{2E}\left[\frac{S_{yt}}{N}\right]^2 \quad \text{--- (4)}$$

Total Strain Energy Theory (T.S.E.T.)

put above values in condⁿ of safe design

$$\sigma_1^2+\sigma_2^2+\sigma_3^2-2\mu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1) \le \left(\frac{S_{yt}}{N}\right)^2$$

for biaxial state of stress, $\sigma_3=0$

$$\sigma_1^2+\sigma_2^2-2\mu\sigma_1\sigma_2 \le \left(\frac{S_{yt}}{N}\right)^2$$

$\Downarrow$

eqⁿ of an ellipse

Semi-major axis $= \dfrac{S_{yt}}{\sqrt{1-\mu}}$

Semi-minor axis $= \dfrac{S_{yt}}{\sqrt{1+\mu}}$

- This theory does not have much significance since it is possible for a material to absorb considerable amount of energy without failure or permanent deformation when it is subjected to hydrostatic pressure.

Maximum Distortion Energy Theory (M.D.E.T.)

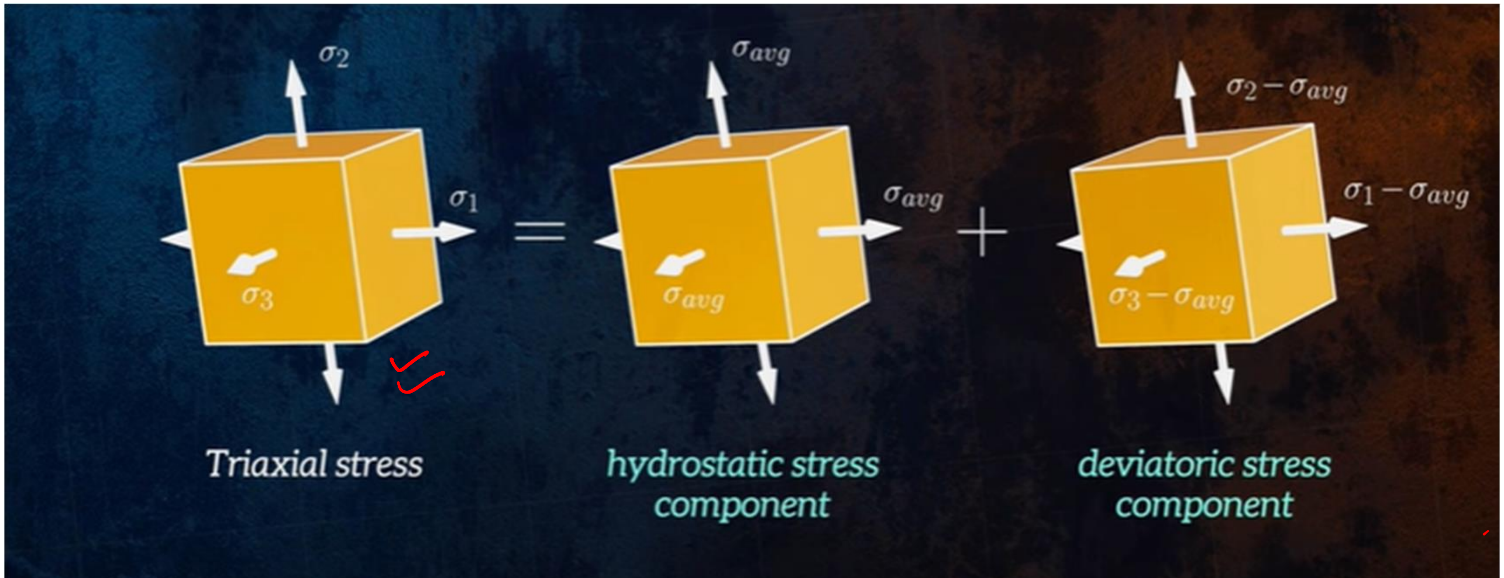
- According to this theory, it is not the total energy which is the criterion for failure; in fact the energy absorbed during the distortion of an element is responsible for failure. The energy of distortion can be obtained by subtracting the energy of volumetric expansion from the total energy.
- Any given state of stress can be uniquely resolved into an isotropic (hydrostatic) state and a pure shear (or deviatoric) state. If σ_1, σ_2 and σ_3 are the principal stresses at a point then

$$\begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_3\end{bmatrix}=\begin{bmatrix}p & 0 & 0\\ 0 & p & 0\\ 0 & 0 & P\end{bmatrix}=\begin{bmatrix}\sigma_1 - p & 0 & 0\\ 0 & \sigma_2 - p & 0\\ 0 & 0 & \sigma_3 - p\end{bmatrix}$$

where $p = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$.

Maximum Distortion Energy Theory (M.D.E.T.)

- **Hydrostatic stress** do not cause yielding in ductile materials while **deviatoric component** is responsible for the yielding.



$$\sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} \sigma_{avg} & 0 & 0 \\ 0 & \sigma_{avg} & 0 \\ 0 & 0 & \sigma_{avg} \end{bmatrix} + \begin{bmatrix} \sigma_1 - \sigma_{avg} & 0 & 0 \\ 0 & \sigma_2 - \sigma_{avg} & 0 \\ 0 & 0 & \sigma_3 - \sigma_{avg} \end{bmatrix}$$

Maximum Distortion Energy Theory (M.D.E.T.)

Cond^n for safe design

$$\text{Maximum D.E./Vol} \le [(\text{D.E./Vol.})_{y.p.}]_{T.T.} \quad \text{—(1)}$$

Total S.E./Vol. = Volumetric S.E./Vol + D.E./Vol.

D.E./Vol. = Total S.E./Vol. − Volum. S.E./Vol.

$$\text{Total S.E./Vol.} = \frac{1}{2E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-2\mu(\sigma_1\sigma_2+\sigma_3\sigma_1+\sigma_2\sigma_3)\right] \quad \text{—(2)}$$

$$\text{Vol. S.E./Vol.} = \frac{1}{2}(\sigma_{avg})\epsilon_v = \frac{1}{2}\left(\frac{\sigma_1+\sigma_2+\sigma_3}{3}\right)\left(\frac{1-2\mu}{E}\right)(\sigma_1+\sigma_2+\sigma_3)$$

$$\therefore \text{Vol. S.E./Vol.} = \left(\frac{1-2\mu}{6E}\right)(\sigma_1+\sigma_2+\sigma_3)^2 \quad \text{—(3)}$$

$$\therefore \text{D.E./Vol.} = \left(\frac{1+\mu}{6E}\right)\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right] \quad \text{—(4)}$$

For $[(\text{D.E./Vol.})_{y.p.}]_{T.T.}$ sub. $\sigma_2=\sigma_3=0$, $\sigma_1=\sigma=\frac{S_{yt}}{N}$ in (4),

$$[(\text{D.E./Vol.})_{y.p.}]_{T.T.} = \left(\frac{1+\mu}{3E}\right)\left(\frac{S_{yt}}{N}\right)^2 \quad \text{—(5)}$$

put (4) & (5) in cond^n of safe design

$$(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2 \le 2\left(\frac{S_{yt}}{N}\right)^2$$

for biaxial state of stress, $\sigma_3=0$

$$\therefore \sigma_1^2+\sigma_2^2-\sigma_1\sigma_2 \le \left(\frac{S_{yt}}{N}\right)^2$$

⇓

Eq^n of ellipse

Semi major Axis $= \sqrt{2}\,S_{yt} = 1.414\,S_{yt}$

Semi minor Axis $= \sqrt{\frac{2}{3}}\,S_{yt} = 0.816\,S_{yt}$

Not valid under Hydrostatic stress condition

A bolt is subjected to an axial pull of 12 kN together with a transverse shear force of 6 kN. Determine the diameter of the bolt by using (i) Maximum principal stress theory (ii) Maximum strain theory (iii) Maximum distortion energy theory (iv) Maximum shear stress theory (Take E_t=300 MPa, fos=3, Poisson's ratio=0.3). **(12.5)**

Ratio of $\frac{S_{YS}}{S_{Yt}}$ by using theories of failure

1. S_{ys} (Yield strength in shear) is obtained from torsion test.

2. Torsion test is conducted under pure torsion i.e. pure shear state of stress ($\sigma_x = \sigma_y = 0$; $\tau_{xy} = \tau$).

3. Under pure shear state of stress

$$\sigma_1 = \tau \, , \, \sigma_2 = -\tau \quad \text{and } \tau = \frac{16T}{\pi d^3}$$

4. S_{ys} can also be obtained by applying theories of failure for pure shear state of stress condition.

5. When yielding in shear occurs under pure shear state of stress, $\tau = S_{ys}$.

(a) $\frac{S_{YS}}{S_{Yt}}$ in Maximum Principal stress theory

According to M.P.S.T,

Considering Factor of safety (N) = 1

$\sigma_1 \leq S_{yt}$ or

$\sigma_1 = S_{yt}$

But in pure shear state of stress, $\sigma_1 = \tau$

$\tau = S_{yt}$

When yielding occurs in shear under pure shear state of stress, $\tau = S_{ys}$

$S_{ys} = S_{yt}$

$\frac{S_{YS}}{S_{Yt}} = 1$

(b) $\frac{S_{YS}}{S_{Yt}}$ in Maximum shear stress theory

According to M.S.S.T,

$|\sigma_1 - \sigma_2| \leq S_{yt}$

But in pure shear state of stress, $\sigma_1 = \tau$ and $\sigma_2 = -\tau$

$\tau - (-\tau) = S_{yt}$

$2\tau = S_{yt}$

When yielding occurs in shear under pure shear state of stress, $\tau = S_{ys}$

$\frac{S_{YS}}{S_{Yt}} = \frac{1}{2}$

(c) $\frac{S_{YS}}{S_{Yt}}$ in Maximum principal strain theory

According to M.P.St.T,

$\sigma_1 - \mu(\sigma_2) = S_{yt}$

$\tau - \mu(-\tau) = S_{yt}$

$\tau(1+\mu) = S_{yt}$

$$S_{ys} = \frac{S_{yt}}{1+\mu}$$

for $\mu = 0.3$

$$\frac{S_{YS}}{S_{Yt}} = 0.77$$

(d) $\frac{S_{YS}}{S_{Yt}}$ in Total strain energy theory

According to T.St.E.T,

$\sigma_1^2 + \sigma_2^2 - 2\mu\,\sigma_1\sigma_2 = S_{yt}^2$

$\tau^2 + \tau^2 + 2\,\tau^2 = S_{yt}^2$

$$\tau = \frac{Syt}{\sqrt{2(1+\mu}}$$

$$S_{ys} = \frac{Syt}{\sqrt{2(1+\mu}}$$

for $\mu = 0.3$

$$\frac{S_{YS}}{S_{Yt}} = 0.62$$

(d) $\frac{S_{YS}}{S_{Yt}}$ in Maximum distortion energy theory

According to M.D.E.T,

$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = S_{yt}^2$

$\tau^2 + \tau^2 + \tau^2 = S_{yt}^2$

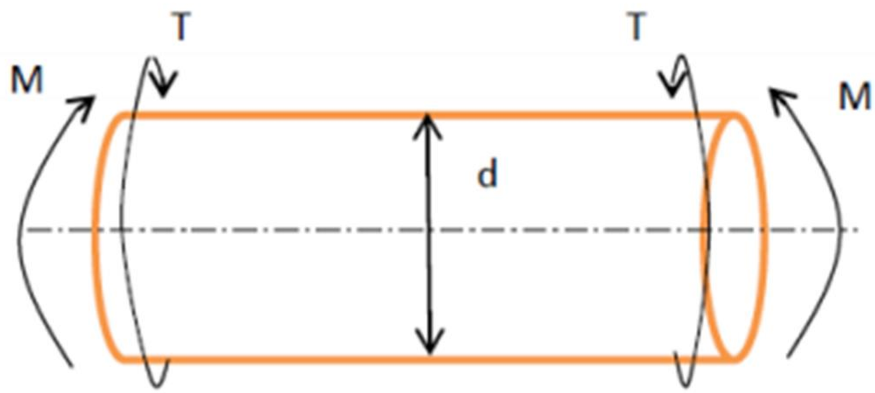
$$\tau = \frac{Syt}{\sqrt{3}}$$

$$S_{ys} = \frac{Syt}{\sqrt{3}}$$

$$\frac{S_{YS}}{S_{Yt}} = 0.577$$

Equivalent Bending Moment (M_e) and Twisting Moment (T_e) equations

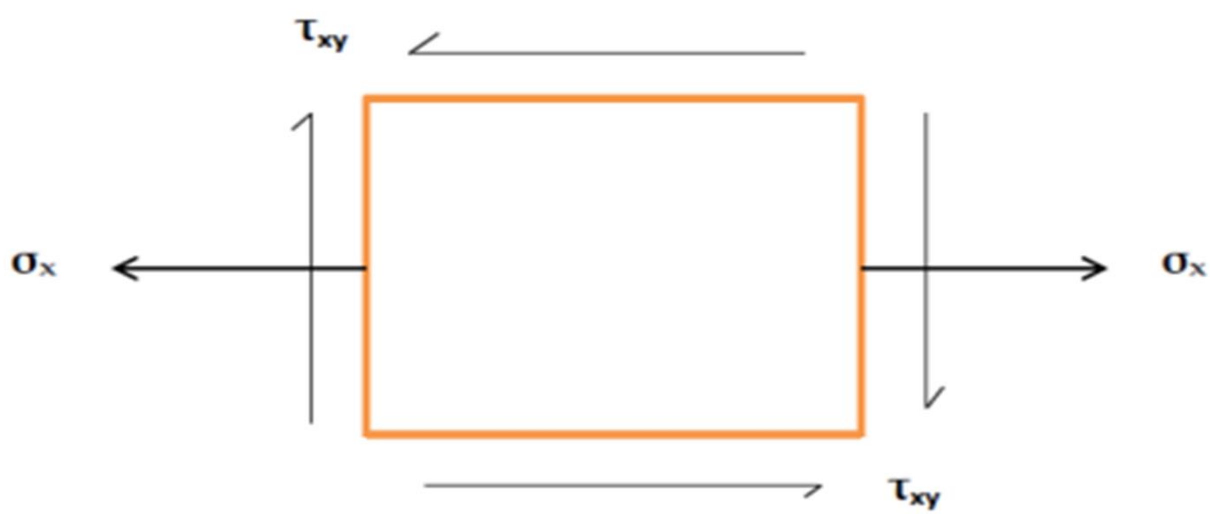
These equations should be used when the component is subjected to both Bending Moment and Twisting Moment simultaneously.



T.O.F	M_e and T_e Equations
M.P.S.T	$M_e = \frac{1}{2}\left[M + \sqrt{M^2 + T^2} \right] = \frac{\pi}{32} d^3 \sigma_{per}$
M.S.S.T	$T_e = \sqrt{M^2 + T^2} = \frac{\pi}{16} d^3 \tau_{per}$
M.D.E.T	$M_e = \sqrt{M^2 + \frac{3}{4} T^2} = \frac{\pi}{32} d^3 \sigma_{per}$

Normal Stress Equations (σ_t equations)

Normal stress equations should be used when a point in a component is subjected to normal stress in one direction only and a shear stress.



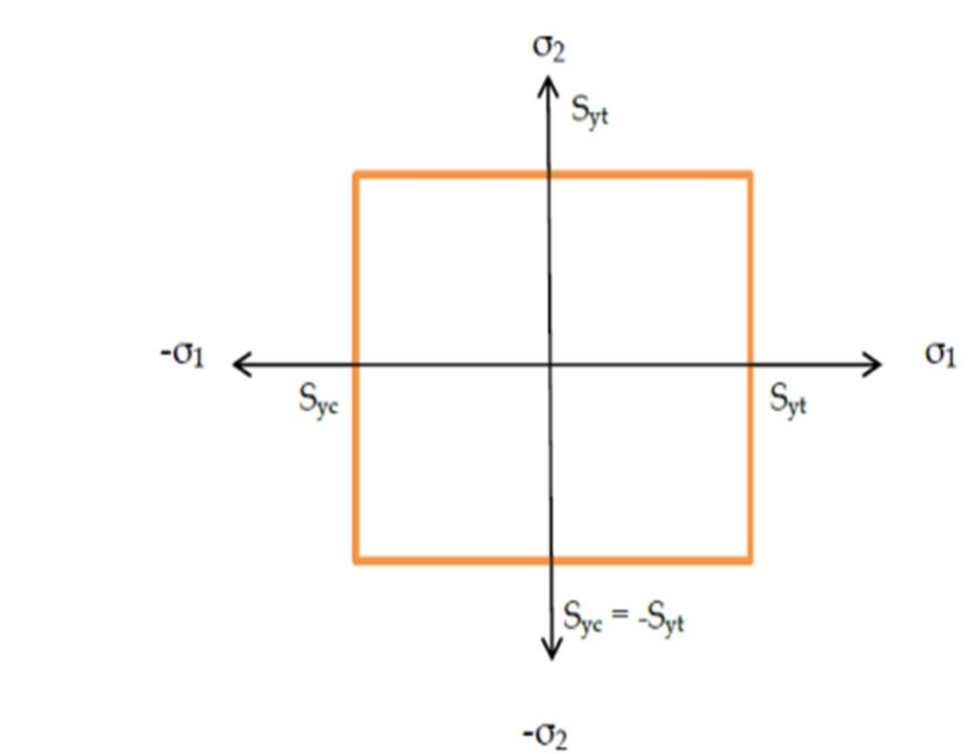
T.O.F	σ_t equations
M.P.S.T	$\sigma_t = \frac{1}{2}\left[\sigma_x + \sqrt{\sigma_x^2 + 4\tau_{xy}^2}\right] = \frac{S_{yt}}{N}$
M.S.S.T	$\sigma_t = \sqrt{\sigma_x^2 + 4\tau_{xy}^2} = \frac{S_{yt}}{N}$
M.D.E.T	$\sigma_t = \sqrt{\sigma_x^2 + 3\tau_{xy}^2} = \frac{S_{yt}}{N}$

Shape of safe boundaries for theories of failure

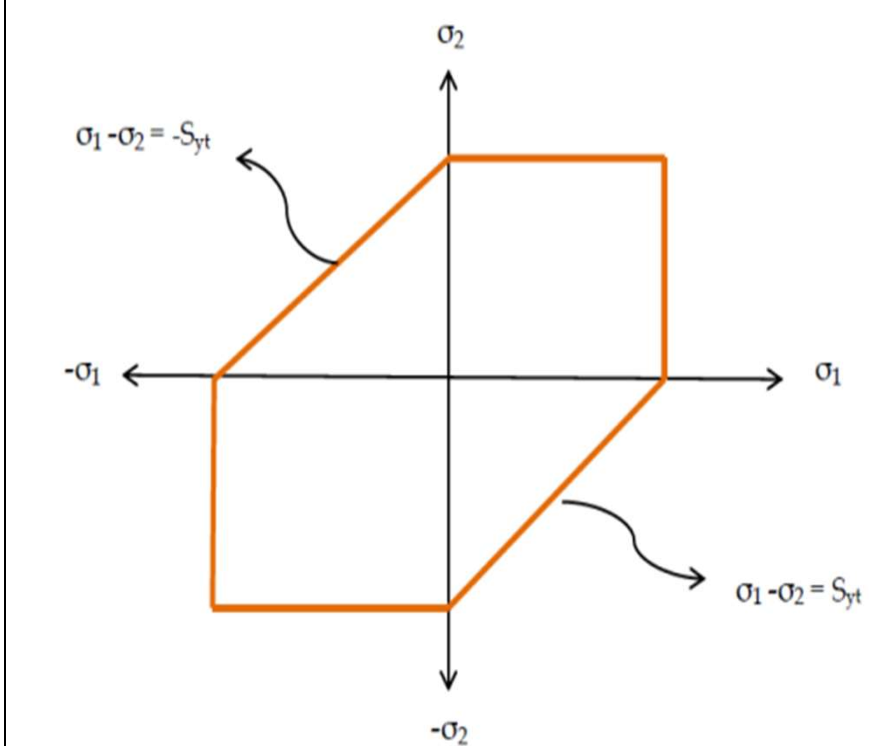
Graphical representation or safe boundaries are used to check whether the given dimensions of a component are safe or not under given loading conditions.

As per theories of failure for ductile material, $S_{yc} = -S_{yt}$

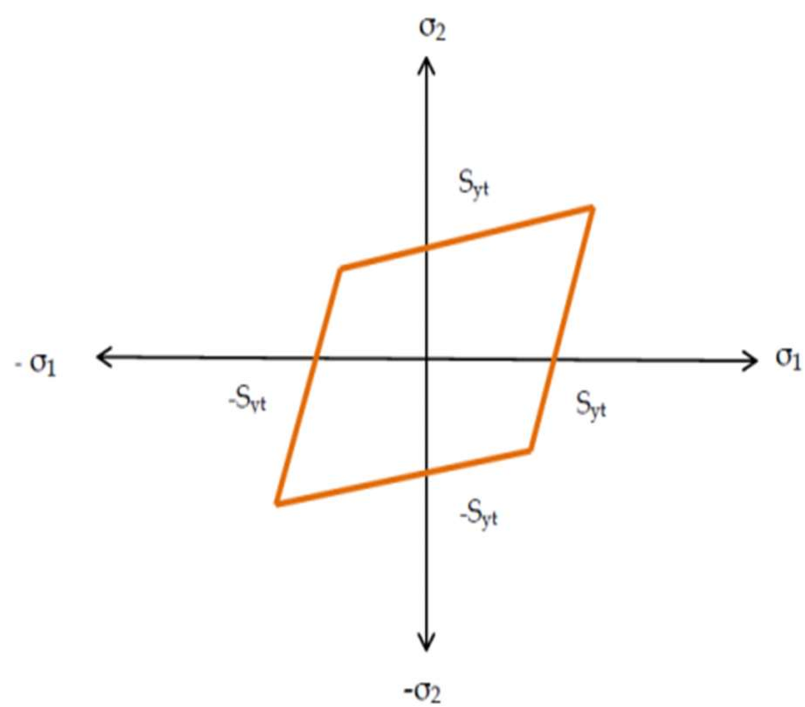
(a) M.P.S.T :- Square



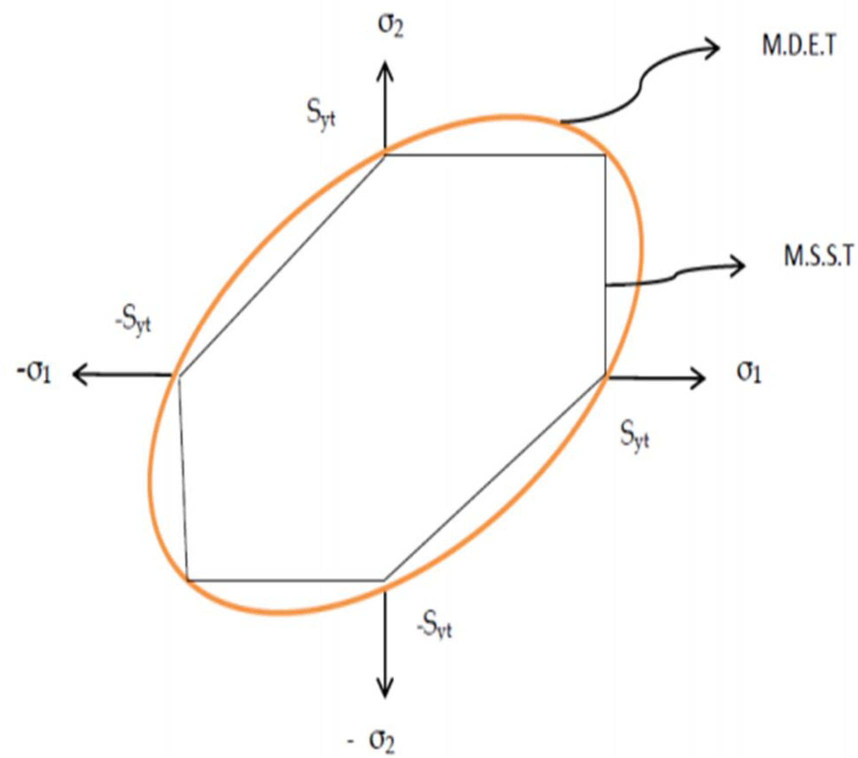
(b) M.S.S.T :- Hexagon



(c) M.P.St.T :- Rhombus



(c) M.D.E.T :- Ellipse



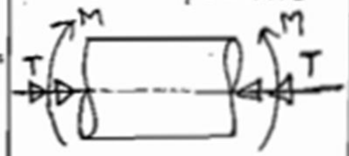
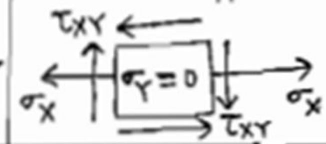
As the area bounded by the curve increases, failure stresses increases thereby decreases dimensions and hence cost of safety.

In all the quadrants

Area bounded by the MDET curve > Aread bounded by MSST curve

Hence

$(\text{Dimensions})_{MDET} < (\text{Dimensions})_{MSST}$

T.O.F	DESIGN EQUATIONS 3D	DESIGN EQUATIONS 2D	Me & Te Equations 	$(\sigma_t)_{per} = \frac{S_{yt}}{N}$ 	$\frac{S_{ys}}{S_{yt}}$	SHAPE OF SAFE BOUNDARY ($S_{yc} = -S_{yt}$)
M.P.S.T RANKINE'S	$\sigma_1 \le \frac{S_{yt}}{N}$ OR $\frac{S_{ut}}{N}$	$\sigma_1 \le \frac{S_{yt}}{N}$ OR $\frac{S_{ut}}{N}$	$M_e = \frac{1}{2}\left[M + \sqrt{M^2+T^2}\right] \le \frac{\pi}{32} d^3 (\sigma_t)_{per}$	$(\sigma_t)_{per} = \frac{1}{2}\left[\sigma_x + \sqrt{\sigma_x^2 + 4\tau_{xy}^2}\right]$	1	SQUARE
M.S.S.T GUEST & TRESCA'S	larger of $\left[\lvert\sigma_1-\sigma_2\rvert, \lvert\sigma_2-\sigma_3\rvert, \lvert\sigma_3-\sigma_1\rvert\right] \le \frac{S_{yt}}{N}$	$\lvert\sigma_1\rvert \le \frac{S_{yt}}{N} \Rightarrow \sigma_{1,2}$ are like in nature $\lvert\sigma_1-\sigma_2\rvert \le \frac{S_{yt}}{N} \Rightarrow \sigma_{1,2}$ are unlike in nature	$T_e = \sqrt{M^2+T^2} \le \frac{\pi}{16} d^3 \tau_{per}$	$(\sigma_t)_{per} = \sqrt{\sigma_x^2 + 4\tau_{xy}^2}$	$\frac{1}{2} = 0.5$	HEXAGON
M.D.E.T VON-MISE'S	$\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right] \le 2\left(\frac{S_{yt}}{N}\right)^2$	$\left[\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2\right] \le \left(\frac{S_{yt}}{N}\right)^2$	$M_e = \sqrt{M^2+\frac{3}{4}T^2} \le \frac{\pi}{32} d^3 (\sigma_t)_{per}$	$(\sigma_t)_{per} = \sqrt{\sigma_x^2 + 3\tau_{xy}^2}$	$\frac{1}{\sqrt{3}} = 0.577$	ELLIPSE Semimajor axis $= \sqrt{2} S_{yt}$ Semiminor axis $= \sqrt{\frac{2}{3}} S_{yt}$
T.S.E.T HAIGH'S	$\sigma_1^2+\sigma_2^2+\sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \le \left(\frac{S_{yt}}{N}\right)^2$	$\sigma_1^2+\sigma_2^2-2\mu\sigma_1\sigma_2 \le \left(\frac{S_{yt}}{N}\right)^2$	—	—	$\frac{1}{\sqrt{2(1+\mu)}} = 0.62$	ELLIPSE semimajor axis $= \frac{S_{yt}}{\sqrt{1-\mu}}$ semiminor axis $= \frac{S_{yt}}{\sqrt{1+\mu}}$
M.P.St.T St.VENANT	$\sigma_1 - \mu(\sigma_2+\sigma_3) \le \frac{S_{yt}}{N}$	$\sigma_1 - \mu\sigma_2 \le \frac{S_{yt}}{N}$	—	—	$\frac{1}{(1+\mu)} = 0.77$	RHOMBUS

Important points

- M_e and T_e equations should be used when a solid circular component is subjected to both BM and TM simultaneously.
- M.P.S.T. is the best T.O.F. for brittle material because they are weak in tension.
- M.D.E.T. is the best T.O.F. for ductile material because it gives safe and economic design.
- M.S.S.T. is suitable for ductile material but it gives over safe design i.e. safe and uneconomical.
- M.S.S.T. and M.D.E.T. are not valid under hydrostatic condition because every plane is principal plane.
- M.P.S.T., M.P.St.T. and T.S.E.T. are suitable under hydrostatic state of stress condition.